ELEMENTARY PROPERTIES OF THE STIELTJES INTEGRAL.*

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This paper contains the proofs of certain elementary properties of the Stieltjes integral. The proofs themselves are elementary in character, in the sense that they do not involve point-set theory to any extent. This fact, and the wide possible application of the Stieltjes integral to subjects in mathematical physics, would seem to justify their collection by themselves.

The first theorem deals with the existence of the Stieltjes integral, and is merely introductory. The others lead up to a theorem on the change of the order of integration of an iterated integral, and to a theorem on integration by parts, the latter having been proved and used extensively by Stieltjes.† It is repeated here for the sake of completeness.

Definition 1. Consider two functions, $\varphi(x)$ and $\alpha(x)$, defined at every point of the interval $a \leq x \leq b$. Let (a, b) be divided by the n points $x_1 < x_2 < x_3 < \cdots < x_n$ into $n + 1$ parts such that $x_{i+1} - x_i = \delta_i \leq \delta$, $i = 0, 1, 2, \cdots, n$, $[x_0 = a, x_{n+1} = b]$, and let the points ξ_i be chosen so that $x_i \leq \xi_i \leq x_{i+1}$. The quantity

$$\lim_{\delta=0} \sum_{i=0}^{n} \varphi(\xi_i) \{\alpha(x_{i+1}) - \alpha(x_i)\},$$

if it exists, is called the *Stieltjes integral of* $\varphi(x)$, *with regard to* $\alpha(x)$, *between the limits a and b. It is designated by* $\int_a^b \varphi(x) d\alpha(x)$.

Theorem 1. If $\alpha(x)$ is a function of limited variation, $a \leq x \leq b$, and if $\varphi(x)$ is continuous at every point of (a, b), then $\int_a^b \varphi(x) d\alpha(x)$ exists.

The proof depends on the fact that, since $\alpha(x)$ is of limited variation,

* The writing of the proofs of these theorems was suggested by Professor G. C. Evans; see Cambridge Colloquium Lectures, Lecture V.

† Stieltjes, Recherches sur les fractions continues, Annales de la Faculté des Sciences de Toulouse, 1894.

The concept of the Stieltjes integral has been extended by Radon, Sitzungsberichte der Akademie der Wissenschaften, Wien, 1913. 122, p. 1295.

we can write

$$\alpha(x) = \alpha(a) + P(x) - N(x),$$

where $P(x)$, $N(x)$ are limited non-decreasing functions.*

Evidently

$$\sum_{i=0}^{n} \varphi(\xi_i)\{\alpha(x_{i+1}) - \alpha(x_i)\} = \sum_{i=0}^{n} \varphi(\xi_i)\{P(x_{i+1}) - P(x_i)\} - \sum_{i=0}^{n} \varphi(\xi_i)\{N(x_{i+1}) - N(x_i)\}.$$

It is sufficient, therefore, to prove the existence of

$$\int_a^b \varphi(x)d\beta(x) = \lim_{\delta=0} \sum_{i=0}^{n} \varphi(\xi_i)\{\beta(x_{i+1}) - \beta(x_i)\},$$

where $\beta(x)$ is limited and non-decreasing in (a, b).

Let $M_i(m_i)$ be the maximum (minimum) value of $\varphi(x)$ in the sub-interval $x_i \leq x \leq x_{i+1}$, and let

$$S_n = \sum_{i=0}^{n} M_i\{\beta(x_{i+1}) - \beta(x_i)\}, \qquad s_n = \sum_{i=0}^{n} m_i\{\beta(x_{i+1}) - \beta(x_i)\}.$$

Evidently

$$S_n \geq s_n \tag{1}$$

if both sums are formed according to the same mode of division of (a, b). Also

$$M\{\beta(b) - \beta(a)\} \geq S_n, \qquad m\{\beta(b) - \beta(a)\} \leq s_n, \tag{2}$$

where $M(m)$ is the maximum (minimum) value of φ in (a, b), for

$$\sum_{i=0}^{n} M\{\beta(x_{i+1}) - \beta(x_i)\} \geq \sum_{i=0}^{n} M_i\{\beta(x_{i+1}) - \beta(x_i)\}.$$

By applying inequalities (2) to subintervals of (a, b) instead of to (a, b) itself, it is easily seen that, if $S_{n_1+n_2}$ is a sum formed by adding n_2 points of division to those of S_{n_1},

$$S_{n_1+n_2} \leq S_{n_1}, \qquad s_{n_1+n_2} \geq s_n\,. \tag{3}$$

* de la Vallée Poussin, Cours d'analyse, Vol. I, third edition, p. 72 et seq. The facts are as follows: Regard the sum $\tau = \sum_{i=0}^{n} |\, \alpha(x_{i+1}) - \alpha(x_i) \,|$, taken over the closed interval (a, x), as made up of terms of two classes, viz.: those corresponding (i) to positive (ii) to negative differences. Let $P(x)$, $N(x)$ be the upper limits, respectively, of the sums of the two classes of terms, for all modes of division of (a, x), and let $T(x)$ be, similarly, the limit of τ. If $T(x)$ exists, $\alpha(x)$ is, by definition, of limited variation in (a, x). The existence of $T(x)$, called the total variation function of $\alpha(x)$, implies that of both $P(x)$ and $N(x)$. $T(x) = P(x) + N(x)$. These three are, of course, non-decreasing functions, and $\alpha(x) = \alpha(a) + P(x) - N(x)$.

We can now show that every S, no matter how formed, is at least as great as every s. For suppose $S_{n_1} < s_{n_2}$; then, by (3),

$$S_{n_1+n_2} \leq S_{n_1} < s_{n_2} \leq s_{n_1+n_2},$$

which is impossible, in view of (1).

The numbers S have a lower limit L, and the numbers s an upper limit l. To show that

$$L = l = \int_a^b \varphi(x) d\beta(x)$$

we have only to note that

$$S_n \geq \sum_{i=0}^{n} \varphi(\xi_i) \{\beta(x_{i+1}) - \beta(x_i)\} \geq s_n$$

and to show that, by taking δ small enough, $S_n - s_n$ can be made as small as we please.

Since $\varphi(x)$ is continuous in a closed interval, and therefore uniformly continuous, it is possible to choose δ so small that, for any ϵ assigned in advance,

$$M_i - m_i \leq \frac{\epsilon}{\beta(b) - \beta(a)} \qquad i = 0, 1, 2, \cdots, n.$$

Hence

$$S_n - s_n = \sum_{i=0}^{n} \{M_i - m_i\} \{\beta(x_{i+1}) - \beta(x_i)\} \leq \epsilon.$$

The theorem is thus proved.

In some of the theorems which follow it is convenient to be able to reduce the proof to the discussion of finite sums instead of integrals. We therefore introduce the following lemma:

Theorem 2.* If $\alpha(s)$ is of limited variation, $a \leq s \leq b$, and if

$$\int_a^b \varphi d\alpha = \lim_{\delta=0} \sum_{i=0}^{n} \varphi(\xi_i) \{\alpha(s_{i+1}) - \alpha(s_i)\}$$

exists, then

$$\left| \int_a^b \varphi d\alpha - \sum_{i=0}^{n} \varphi(\xi_i) \{\alpha(s_{i+1}) - \alpha(s_i)\} \right| \leq O_\delta T(b)$$

where $T(b)$ is the total variation of α in the interval (a, b), O_δ is the greatest oscillation of $\varphi(s)$ in the $n + 1$ intervals (s_i, s_{i+1}), and $s_i \leq \xi_i \leq s_{i+1}$.

Consider the sum

$$\sum_{i=0}^{n'} \varphi(\xi_i') \{\alpha(s_{i+1}') - \alpha(s_i')\},$$

formed by a set of points (s_i') which includes the set (s_i).

* This proof of Theorem 2 was suggested by Mr. H. L. Smith and is more ingenious and somewhat shorter than that of the writer, for which it is therefore substituted.

Evidently

$$\left|\int_a^b \varphi d\alpha - \sum_{i=0}^{n} \varphi(\xi_i)\{\alpha(s_{i+1}) - \alpha(s_i)\}\right|$$

$$\leq \left|\int_a^b \varphi(d\alpha) - \sum_{i=0}^{n'} \varphi(\xi_i')\{\alpha(s_{i+1}') - \alpha(s_i')\}\right|$$

$$+ \left|\sum_{i=0}^{n} \varphi(\xi_i)\{\alpha(s_{i+1}) - \alpha(s_i)\} - \sum_{i=0}^{n'} \varphi(\xi_i')\{\alpha(s_{i+1}') - \alpha(s_i')\}\right|.$$

But by choosing a small enough norm for the points s_i' the first term of the right-hand member, by our hypothesis, can be made as small as we please, and, since the second term is at most equal to $O_\delta \int |d\alpha| = O_\delta T(b)$,

$$\left|\int_a^b \varphi d\alpha - \sum_{i=0}^{n} \varphi(\xi_i)\{\alpha(s_{i+1}) - \alpha(s_i)\}\right| \leq O_\delta T(b).$$

Definition 2. Consider a function $\alpha(x, s)$ which, for every value of x, $c \leq x \leq d$, is of limited variation in s, $a \leq s \leq b$. Let $T(x, s)$ be the corresponding total variation function in s, for a given value of x. Evidently $T(x, s) \leq T(x, b)$.

If $T(x, b)$ is a limited function of x, i. e., if a positive constant K can be found, such that

$$T(x, b) < K \qquad c \leq x \leq d$$

then $\alpha(x, s)$ is said to be a *function of uniformly limited variation in s for all values of x in (c, d)*.

Theorem 3. If $\varphi(s)$ is continuous, $a \leq s \leq b$, and if $\alpha(x, s)$ is a function of uniformly limited variation in s for all values of x in the interval $c \leq x \leq d$, and if at every point x in (c, d), a set of values of s,* dense in (a, b) and including a and b, can be found, such that for each value of s in the set $\alpha(x, s)$ is continuous in x, then $\Phi(x) = \int_a^b \varphi(s) d_s\alpha(x, s)$ is continuous, $c \leq x \leq d$.

By Theorem 2, a positive constant K exists such that

$$\left|\Phi(x) - \sum_{i=0}^{n} \varphi(\xi_i)\{\alpha(x, s_{i+1}) - \alpha(x, s_i)\}\right| \leq O_\delta K$$

and

$$\left|\Phi(x + \Delta x) - \sum_{i=0}^{n} \varphi(\xi_i)\{\alpha(x + \Delta x, s_{i+1}) - \alpha(x + \Delta x, s_i)\}\right| \leq O_\delta K.$$

* This set of values of s need not of course be independent of x.

Therefore

$$\left|[\Phi(x+\Delta x)-\Phi(x)]-\left[\sum_{i=0}^{n}\varphi(\xi_i)\{\alpha(x+\Delta x, s_{i+1})-\alpha(x+\Delta x, s_i)\}\right.\right.$$
$$\left.\left.-\sum_{i=0}^{n}\varphi(\xi_i)\{\alpha(x, s_{i+1})-\alpha(x, s_i)\}\right]\right| \leq 2O_\delta K.$$

Now, if ϵ be assigned, a norm δ can be found, so small that, since $\varphi(s)$ is uniformly continuous, $O_\delta < \frac{1}{4}\cdot\frac{\epsilon}{K}$ and therefore

$$|\Phi(x+\Delta x)-\Phi(x)| \leq \tfrac{1}{2}\epsilon + \left|\sum_{i=0}^{n}\varphi(\xi_i)\{\alpha(x+\Delta x, s_{i+1})-\alpha(x+\Delta x, s_i)\}\right.$$
$$\left.-\sum_{i=0}^{n}\varphi(\xi_i)\{\alpha(x, s_{i+1})-\alpha(x, s_i)\}\right|.$$

This inequality holds for all modes of division of the interval $a \leq s \leq b$ of norm δ. Let us select a mode, of norm δ, such that the points of division, $a, s_1, s_2, \cdots, s_n, b$ are all points at which $\alpha(x, s)$ is continuous in x. Then the term between absolute value signs is merely the difference of the continuous function of x, $\sum_{i=0}^{n}\varphi(\xi_i)\{\alpha(x, s_{i+1})-\alpha(x, s_i)\}$, taken at the values x and $x+\Delta x$, and therefore can be made as small as we please ($< \epsilon/2$) by taking Δx small enough; whence it follows that

$$|\Phi(x+\Delta x)-\Phi(x)| < \epsilon$$

and the theorem is proved.

Theorem 4. If $\gamma(x)$ is a function of limited variation, $c \leq x \leq d$, and if $\alpha(x, s)$ is of uniformly limited variation in s, $a \leq s \leq b$, for every value of x, $c \leq x \leq d$, and continuous in x for every value of s, then

$$\Psi(s) = \int_c^d \alpha(x, s)d\gamma(x)$$

is a function of limited variation, $a \leq s \leq b$.

It is required to show that a positive constant L can be found, such that, for any mode of division whatever of the interval (a, b), by points $s_0 = a, s_1, s_2, \cdots, s_m, s_{m+1} = b$,

$$\sum_{j=0}^{m}|\Psi(s_{j+1})-\Psi(s_j)| \leq L.$$

By expressing $\gamma(x)$ as the difference of two limited, non-decreasing functions, $\gamma_1(x)$ and $\gamma_2(x)$, we have:

$$\Psi(s) = \int_c^d \alpha(x, s)d\gamma_1(x) - \int_c^d \alpha(x, s)d\gamma_2(x),$$

and since the difference of two functions of limited variation is also of limited variation, our theorem will be proved if we can show that

$$\Psi(s) = \int_c^d \alpha(x, s) d\bar{\gamma}(x)$$

is of limited variation in the special case in which $\bar{\gamma}(x)$ is a limited, non-decreasing function. It will be shown that

$$\sum_{j=0}^{m} |\Psi(s_{j+1}) - \Psi(s_j)| \leq 2K[\bar{\gamma}(d) - \bar{\gamma}(c)],$$

where K, as before, is the upper bound of the total variation of $\alpha(x, s)$.

Regarding the quantities $s_1, s_2, \cdots, s_m$, as fixed, and applying the approximation formula of Theorem 2, we obtain

$$\left|\Psi(s_j) - \sum_{i=0}^{n} \alpha(\xi_i, s_j)\{\bar{\gamma}(x_{i+1}) - \bar{\gamma}(x_i)\}\right| \leq O_\delta^{[j]}[\bar{\gamma}(d) - \bar{\gamma}(c)]$$
$$j = 0, 1, 2, \cdots (m+1),$$

where $O_\delta^{[j]}$ is the greatest oscillation of $\alpha(x, s_j)$ in any of the intervals $x_i \leq x \leq x_{i+1}$, $i = 0, 1, 2, \cdots, n$.

Since $\alpha(x, s)$ is continuous in x for each of the values $s = a, s_1, s_2, \cdots, s_m, b$, a single mode of division of the interval $c \leq x \leq d$ can be chosen, of norm δ so small that

$$O_\delta^{[j]} < \frac{K}{2(m+1)},$$

this inequality holding, at once, for all of the given values of s_j. Hence

$$\begin{aligned} |\Psi(s_{j+1}) - \Psi(s_j)| \leq & \left|\sum_{i=0}^{n} \{\alpha(\xi_i, s_{j+1}) - \alpha(\xi_i, s_j\}\{\bar{\gamma}(x_{i+1}) - \bar{\gamma}(x_i)\}\right| \\ & + \frac{K}{m+1}[\bar{\gamma}(d) - \bar{\gamma}(c)] \qquad j = 0, 1, 2, \cdots, m. \end{aligned} \tag{1}$$

Now

$$\begin{aligned} &\left|\sum_{i=0}^{n} \{\alpha(\xi_i, s_{j+1}) - \alpha(\xi_i, s_j)\}\{\bar{\gamma}(x_{i+1}) - \bar{\gamma}(x_i)\}\right| \\ &= \left|\sum_{i=0}^{n} \{P(\xi_i, s_{j+1}) - N(\xi_i, s_{j+1}) - P(\xi_i, s_j) + N(\xi_i, s_j)\}\{\bar{\gamma}(x_{i+1}) - \bar{\gamma}(x_i)\}\right| \\ &\leq \left|\sum_{i=0}^{n} \{P(\xi_i, s_{j+1}) + N(\xi_i, s_{j+1}) - P(\xi_i, s_j) - N(\xi_i, s_j)\}\{\bar{\gamma}(x_{i+1}) - \bar{\gamma}(x_i)\}\right|, \end{aligned}$$

since

$$N(\xi_i, s_{j+1}) \geq N(\xi_i, s_j),$$

and this last expression can be written:

$$\sum_{i=0}^{n} \{T(\xi_i, s_{j+1}) - T(\xi_i, s_j)\}\{\bar{\gamma}(x_{i+1}) - \bar{\gamma}(x_i)\}.$$

Substituting this result in (1) and adding together the results for $j = 0, 1, 2, \cdots, m$, we obtain

$$\sum_{j=0}^{m} |\Psi(s_{j+1}) - \Psi(s_j)| \leq \sum_{j=0}^{m}\sum_{i=0}^{n} \{T(\xi_i, s_{j+1}) - T(\xi_i, s_j)\}\{\bar{\gamma}(x_{i+1}) - \bar{\gamma}(x_i)\} + K[\bar{\gamma}(d) - \bar{\gamma}(c)].$$

Changing the order of summation in the double sum, we obtain

$$\begin{aligned}\sum_{j=0}^{m} |\Psi(s_{j+1}) - \Psi(s_j)| &\leq \sum_{i=0}^{n} \{T(\xi_i, b) - T(\xi_i, a)\}\{\bar{\gamma}(x_{i+1}) - \bar{\gamma}(x_i)\} + K[\bar{\gamma}(d) - \bar{\gamma}(c)] \\ &\leq K\sum_{i=0}^{n} \{\bar{\gamma}(x_{i+1}) - \bar{\gamma}(x_i)\} + K\{\bar{\gamma}(d) - \bar{\gamma}(c)\}\end{aligned}$$

since α is of uniformly limited variation in s. Hence finally

$$\sum_{j=0}^{m} |\Psi(s_{j+1}) - \Psi(s_j)| \leq 2K[\bar{\gamma}(d) - \bar{\gamma}(c)].$$

Theorem 5. If $\varphi(s)$ is continuous, $a \leq s \leq b$; if $\bar{\gamma}(x)$ is of limited variation, $c \leq x \leq d$; and if $\alpha(x, s)$ is continuous in x for all values of s in (a, b) and of uniformly limited variation in s for all values of x in (c, d), then the integrals

$$\int_c^d \left[\int_a^b \varphi(s)d_s\alpha(x, s)\right]d\gamma(x), \qquad \int_a^b \varphi(s)d_s\int_c^d \alpha(x, s)d\gamma(x)$$

exist and are equal.

By Theorem 3 $\Phi(x) = \int_a^b \varphi(s)d_s\alpha(x, s)$ is continuous, and, by Theorem 4, $\Psi(s) = \int_c^d \alpha(x, s)d\gamma(x)$ is of limited variation. Hence, by Theorem 1, $\int_c^d \Phi(x)d\gamma(x)$ and $\int_a^d \varphi(s)d\Psi(s)$ exist.

It is to be shown that, no matter how small a positive constant ϵ be assigned,

$$\left|\int_c^d \Phi(x)d\gamma(x) - \int_a^b \varphi(s)d\Psi(s)\right| < \epsilon.$$

The following relation, which is easily verified, will be used:

$$\begin{aligned}\sum_{j=0}^{m}\left[\sum_{i=0}^{n} \varphi(s_i)\{\alpha(x_j, s_{i+1}) - \alpha(x_j, s_i)\}\right][\gamma(x_{j+1}) - \gamma(x_j)] \\ = \sum_{i=0}^{n} \varphi(s_i)\left[\sum_{j=0}^{m} \alpha(x_j, s_{i+1})\{\gamma(x_{j+1}) - \gamma(x_j)\}\right. \\ \left. - \sum_{j=0}^{m} \alpha(x_j, s_i)\{\gamma(x_{j+1}) - \gamma(x_j)\}\right].\end{aligned} \tag{1}$$

Since $\Phi(x)$ is continuous, $c \leq x \leq d$, it is possible to divide (c, d) into $m + 1$ parts, of norm δ_x so small that, by Theorem 2,

$$\left| \int_c^d \Phi(x) d\gamma(x) - \sum_{j=0}^{m} \Phi(x_j) \{\gamma(x_{j+1}) - \gamma(x_j)\} \right| < \frac{\epsilon}{4}. \tag{2}$$

It is also possible to divide (a, b) into $n + 1$ parts, of norm δ_s so small that

$$\begin{aligned} \Bigg| \sum_{j=0}^{m} \Phi(x_j) \{\gamma(x_{j+1}) - \gamma(x_j)\} - \sum_{j=0}^{m} \Bigg[\sum_{i=0}^{n} \varphi(s_i) \{\alpha(x_j, s_{i+1}) \\ - \alpha(x_j, s_i)\} \Bigg] [\gamma(x_{j+1}) - \gamma(x_j)] \Bigg| < \frac{\epsilon}{4}. \end{aligned} \tag{3}$$

In fact, since $\varphi(s)$ is uniformly continuous in (a, b) and $\alpha(x, s)$ is of uniformly limited variation in s for every x, by Theorem 2, δ_s can be chosen so that, for every x,

$$\left| \Phi(x) - \sum_{i=0}^{n} \varphi(s_i) \{\alpha(x, s_{i+1}) - \alpha(x, s_i)\} \right| < \frac{\epsilon/4}{\tau(d)},$$

where $\tau(d)$ is the total variation of $\gamma(x)$ in (c, d).

From (2) and (3) it follows that, for norms δ_x and δ_s,

$$\begin{aligned} \Bigg| \int_c^d \Phi(x) d\gamma(x) - \sum_{j=0}^{m} \Bigg[\sum_{i=0}^{n} \varphi(s_i) \{\alpha(x_j, s_{i+1}) \\ - \alpha(x_j, s_i)\} \Bigg] [\gamma(x_{j+1}) - \gamma(x_j)] \Bigg| < \frac{\epsilon}{2}, \end{aligned} \tag{4}$$

and by virtue of the uniformity referred to above, (4) is true, a fortiori, for any smaller norms.

Now let (a, b) be divided, norm $\delta_s'(< \delta_s)$, into $n' + 1$ parts so that

$$\begin{aligned} \Bigg| \int_a^b \varphi(s) d_s \int_c^d \alpha(x, s) d\gamma(x) - \sum_{i=0}^{n'} \varphi(s_i) \Bigg[\int_c^d \alpha(x, s_{i+1}) d\gamma(x) \\ - \int_c^d \alpha(x, s_i) d\gamma(x) \Bigg] \Bigg| < \frac{\epsilon}{4} \end{aligned} \tag{5}$$

and let (c, d) be divided, norm $\delta_x'(< \delta_x)$, into $m' + 1$ parts so that

$$\begin{aligned} \Bigg| \sum_{i=0}^{n'} \varphi(s_i) \Bigg[\int_c^d \alpha(x, s_{i+1}) d\gamma(x) - \int_c^d \alpha(x, s_i) d\gamma(x) \Bigg] \\ - \sum_{i=0}^{n'} \varphi(s_i) \Bigg[\sum_{j=0}^{m'} \alpha(x_j, s_{i+1}) \{\gamma(x_{j+1}) - \gamma(x_j)\} - \sum_{j=0}^{m'} \alpha(x_j, s_i) \{\gamma(x_{j+1}) \\ - \gamma(x_j)\} \Bigg] \Bigg| < \frac{\epsilon}{4}. \end{aligned} \tag{6}$$

This can be done, for δ_x' can be chosen so small that

$$\left| \int_c^d \alpha(x, s_i) d\gamma(x) - \sum_{j=0}^{m'} \alpha(x_j, s_i) \{ \gamma(x_{j+1}) - \gamma(x_j) \} \right| < \frac{\epsilon}{8M(n'+1)}$$
$$i = 0, 1, 2, \cdots, n',$$

where $M > |\varphi(s)|$. Hence

$$\left| \left[\int_c^d \alpha(x, s_{i+1}) d\gamma(x) - \int_c^d \alpha(x, s_i) d\gamma(x) \right] - \left[\sum_{j=0}^{m'} \alpha(x_j, s_{i+1}) \{ \gamma(x_{j+1}) - \gamma(x_j) \} - \sum_{j=0}^{m'} \alpha(x_j, s_i) \{ \gamma(x_{j+1}) - \gamma(x_j) \} \right] \right| < \frac{\epsilon}{4M(n'+1)}.$$

From (5) and (6) it follows that, for norms δ_x' and δ_s',

$$\left| \int_a^b \varphi(s) d\Psi(s) - \sum_{i=0}^{n'} \varphi(s_i) \left[\sum_{j=0}^{m'} \alpha(x_j, s_{i+1}) \{ \gamma(x_{j+1}) - \gamma(x_j) \} - \sum_{j=0}^{m'} \alpha(x_j, s_i) \{ \gamma(x_{j+1}) - \gamma(x_j) \} \right] \right| < \frac{\epsilon}{2}.$$

From (4) it follows, since $\delta_x' < \delta_x$ and $\delta_s' < \delta_s$, that

$$\left| \int_c^d \Phi(x) d\gamma(x) - \sum_{j=0}^{m'} \left[\sum_{i=0}^{n'} \varphi(s_i) \{ \alpha(x_j, s_{i+1}) - \alpha(x_j, s_i) \} \right] \times [\gamma(x_{j+1}) - \gamma(x_j)] \right| < \frac{\epsilon}{2}.$$

and, using equation (1),

$$\left| \int_c^d \Phi(x) d\gamma(x) - \int_a^b \varphi(s) d\Psi(s) \right| < \epsilon.$$

Theorem 6. If $\varphi(x)$ is continuous, $a \leq x \leq b$, and if $\alpha(x)$ is of limited variation in the same interval, then the integral $\int_a^b \alpha(x) d\varphi(x)$ exists and is equal to

$$\alpha(x)\varphi(x) \Big|_a^b - \int^b \varphi(x) d\alpha(x).$$

Consider the sum:

$$\sum_{i=0}^{n} \alpha(\xi_i) \{ \varphi(x_{i+1}) - \varphi(x_i) \}$$
$$= \alpha(\xi_0) \{ \varphi(x_1) - \varphi(a) \} + \alpha(\xi_1) \{ \varphi(x_2) - \varphi(x_1) \} + \cdots$$
$$+ \alpha(\xi_{n-1}) \{ \varphi(x_n) - \varphi(x_{n-1}) \} + \alpha(\xi_n) \{ \varphi(b) - \varphi(x_n) \},$$

where $x_i \leq \xi_i \leq x_{i+1}$, $x_{i+1} - x_i \leq \delta$, $x_0 = a$, $x_{n+1} = b$ and also, of course, $\xi_i \leq x_{i+1} \leq \xi_{i+1}$.

By adding and subtracting $\alpha(b)\varphi(b) - \alpha(a)\varphi(a)$ and regrouping the terms we obtain

$$\sum_{i=0}^{n} \alpha(\xi_i)\{\varphi(x_{i+1}) - \varphi(x_i)\} = -\varphi(a)\{(\xi_0) - \alpha(a)\} - \varphi(x_1)\{\alpha(\xi_1) - \alpha(\xi_0)\}$$
$$- \cdots - \varphi(x_n)\{\alpha(\xi_n) - \alpha(\xi_{n-1})\} - \varphi(b)\{\alpha(b) - \alpha(\xi_n)\}$$
$$+ \alpha(b)\varphi(b) - \alpha(a)\varphi(a).$$
$$= \alpha(x)\varphi(x)\Big|_a^b - \sum_{i=0}^{n+1} \varphi(x_i)\{\alpha(\xi_i) - \alpha(\xi_{i-1})\}.$$

In this last summation ξ_{-1} denotes the point a, and ξ_{n+1} the point b.

But

$$\lim_{\delta=0} \sum_{i=0}^{n+1} \varphi(x_i)\{\alpha(\xi_i) - \alpha(\xi_{i-1})\} = \int_a^b \varphi(x)d\alpha(x)$$

for $\xi_i - \xi_{i-1} \leq 2\delta$. Hence

$$\lim_{\delta=0} \sum_{i=0}^{n} \alpha(\xi_i)\{\varphi(x_{i+1}) - \varphi(x_i)\} = \int_a^b \alpha(x)d\varphi(x)$$

exists and is equal to

$$\alpha(x)\varphi(x)\Big|_a^b - \int_a^b \varphi(x)d\alpha(x).$$

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